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Numerical solution of a fourth-order ordinary differential equation

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SUMMARY
In this paper we have derived numerical methods of order $O(h^4)$ and $O(h^6)$ for the solution of a fourth-order ordinary differential equation by finite differences. A method of $O(h^2)$ was earlier discussed by Usmani and Marsden [6]. Convergence of the fourth-order method is shown. Two examples are computed to show the superiority of our methods.

1. Introduction

Consider the boundary value problem

$$y^{\text{iv}} + f(x)y(x) = g(x),\ f(x) \geq 0,\ x \in [a, b] \tag{1}$$

subject to the conditions

$$y(a) = \alpha_1,\ y(b) = \alpha_2;\ y''(a) = \beta_1,\ y''(b) = \beta_2. \tag{2}$$

A particular case of this differential equation gives rise to the problem of bending of a uniformly loaded rectangular plate supported over the entire surface by an elastic foundation and supported rigidly along the edges [5, p. 30]. Problems of this type often occur in plate deflection theory. The analytical solution of (1)–(2) for all $f(x)$ and $g(x)$ cannot always be found. Hence in such situations we have to make use of numerical methods and obtain an approximation to the solution which ensures a desired accuracy. One such method is the one based on finite differences by which the values of y are approximated over a finite set of grid points $x_n \in [a, b]$. Techniques of this type for the solution of ordinary differential equations have been developed by many authors [2, 3, 7]. Recently Usmani and Marsden [6] devised a difference scheme which gave a method of order two for the solution of the problem (1)–(2).

In this paper we have obtained two methods—one of order four and the other of order six—making use of quadrature. We have also solved two examples to illustrate the superiority of these methods in the solution of problems of the type (1)–(2).

2. Difference scheme

We divide the interval $[a, b]$ into a finite set of grid points $x_n = a + nh$, $n = 0(1)N$, where $Nh = b - a$ and denote by y_n the approximation to the value of $y(x)$ at $x = x_n$.

Consider the identity

$$\delta^4 y(x_n) \equiv \frac{1}{6}\left\{\int_{x_n}^{x_{n+2}} (x_{n+2} - t)^3[y^{iv}(t) + y^{iv}(2x_n - t)]\,dt - 4\int_{x_n}^{x_{n+1}} (x_{n+1} - t)^3[y^{iv}(t) + y^{iv}(2x_n - t)]\right\}dt, \qquad n = 2(1)N-2. \tag{3}$$

By using the transformations $t = x_n + h(1 + u)$ in the first integral and $t = x_n + (h/2)(1 + u)$ in the second integral on the right-hand side, (3) can be changed into

$$\delta^4 y(x_n) \equiv \frac{h^4}{6}\int_{-1}^{1} (1-u)^3\left\{y^{iv}[x_n - h(1+u)] + y^{iv}[x_n + h(1+u)] - \tfrac{1}{4}y^{iv}\left[x_n - \frac{h}{2}(1+u)\right] - \tfrac{1}{4}y^{iv}\left[x_n + \frac{h}{2}(1+u)\right]\right\}du. \tag{4}$$

With the aid of suitable weight functions $w(u)$, [4], the integral on the right-hand side of (4) can be evaluated as

$$w_0 y_n^{iv} + w_1[y_{n-1}^{iv} + y_{n+1}^{iv}] + w_2[y_{n-2}^{iv} + y_{n+2}^{iv}] + \sum_{i=1}^{p} w_{r_i}[y_{n-r_i}^{iv} + y_{n+r_i}^{iv} - \tfrac{1}{4}y_{n-r_i/2}^{iv} - \tfrac{1}{4}y_{n+ri/2}^{iv}] + E, \tag{5}$$

where $-2 < r_i < 2$ are the abcissae; w_0, w_1, w_2 and w_{r_i} are the weights and E is the error of the quadrature rule used. The resulting algorithm is

$$\delta^4 y_n = h^4\{w_0 y_n^{iv} + w_1[y_{n-1}^{iv} + y_{n+1}^{iv}] + w_2[y_{n-2}^{iv} + y_{n+2}^{iv}] + \sum_{i=1}^{p} w_{r_i}[y_{n-r_i}^{iv} + y_{n+r_i}^{iv} - \tfrac{1}{4}y_{n-r_i/2}^{iv} - \tfrac{1}{4}y_{n+r_i/2}^{iv}]\}. \tag{6}$$

Thus we see that we can have an algorithm for every choice of the quadrature rule and for every set of parameters selected in the right hand side of (6).

i) If we choose $w_0 = 1$ and $w_1 = 0 = w_2 = w_{r_i}$ for all i, we get the scheme

$$\delta^4 y_n = h^4 y_n^{iv}, \qquad n = 2(1)N-2, \tag{7}$$

used by Usmani and Marsden [6] for developing a second-order method.

ii) If $w_2 = 0 = w_{r_i}$ for all i, we obtain the unique fourth-order scheme

$$\delta^4 y_n = \frac{h^4}{6}[y_{n-1}^{iv} + 4y_n^{iv} + y_{n+1}^{iv}], \qquad n = 2(1)N-2 \tag{8}$$

with truncation error $-(1/720)h^8 y^{(8)}(x_n) + \ldots$. This scheme is unique because any formula involving an off-step point can ultimately be reduced to (8), for

$$y^{iv}_{n-r} + y^{iv}_{n+r} = 2(1 - r^2)y^{iv}_n + r^2[y^{iv}_{n-1} + y^{iv}_{n+1}] + O(h^4). \tag{9}$$

iii) If we take $w_{r_i} = 0$ for all i in the algorithm (6), we get the unique sixth-order scheme

$$\delta^4 y_n = \frac{-h^4}{720}[(y^{iv}_{n-2} + y^{iv}_{n+2}) - 124(y^{iv}_{n-1} + y^{iv}_{n+1}) - 474 y^{iv}_n], \qquad n = 2(1)N-2. \tag{10}$$

As in (ii) above, we may state here that, since

$$y^{iv}_{n-r} + y^{iv}_{n+r} = \frac{(r^2-1)(r^2-4)}{2} y^{iv}_n + \frac{r^2(4-r^2)}{3}[y^{iv}_n + y^{iv}_{n+1}]$$
$$+ \frac{r^2(r^2-1)}{12}[y^{iv}_{n-2} + y^{iv}_{n+2}] + O(h^6), \tag{11}$$

any algorithm of order six depending on off-step points can be reduced to (10), so that (10) is the only scheme of order six depending on five consecutive mesh points.

We note that the system (8) gives us $N-3$ equations for the $N-1$ unknowns y_i, $i = 1(1)N-1$. From the boundary conditions we can get two more relations

$$5y_1 - 4y_2 + y_3 = 2\alpha_1 - h^2\beta_1 + \frac{h^4}{360}[28y^{iv}_0 + 245y^{iv}_1 + 56y^{iv}_2 + y^{iv}_3] \tag{12}$$

and

$$y_{N-3} - 4y_{N-2} + 5y_{N-1}$$
$$= 2\alpha_2 - h^2\beta_2 + \frac{h^4}{360}[y^{iv}_{N-3} + 56y^{iv}_{N-2} + 245y^{iv}_{N-1} + 28y^{iv}_N]. \tag{13}$$

Therefore, the equations (12), (8) and (13) form our method of order four. The equation (10) is of sixth order and to retain the band width of the coefficient matrix A as five, we use the equations (12) and (13) for $n = 1$ and $N-1$ respectively. The truncation error made in these two equations is of order $O(h^4)$. However, numerical results suggest that the method behaves like a method of sixth order, since the $O(h^4)$ term in the error E will have a small coefficient.

Let $\boldsymbol{Y} = (y_n)^T$, $n = 1(1)N-1$. From the equation (1) we get

$$y^{iv}_n = -f_n y_n + g_n, \qquad n = 0(1)N, \tag{14}$$

where $f_n = f(x_n)$ and $g_n = g(x_n)$. If we substitute for y^{iv}_n from (14) in the equations (12), (8) and (13), the system of equations can be written in matrix form as

$$A\boldsymbol{Y} = \boldsymbol{R} \tag{15}$$

where A is a five-band matrix such that

$$a_{12} = -4 + \tfrac{7}{45}h^4 f_2, \qquad a_{13} = 1 + \frac{h^4}{360} f_3,$$

$$a_{N-1,N-2} = -4 + \tfrac{7}{45}h^4 f_{N-2}, \qquad a_{N-1,N-3} = 1 + \frac{h^4}{360} f_{N-3},$$

$$a_{ij} = \begin{cases} 5 + \tfrac{49}{72}h^4 f_i, & i = j = 1, N-1, \\ 6 + \tfrac{2}{3}h^4 f_i, & i = j = 2, 3, \ldots, N-2, \\ -4 + \dfrac{h^4}{6} f_{i-1}, & i - j = 1,\ i = 2, 3, \ldots, N-2, \\ -4 + \dfrac{h^4}{6} f_{i+1}, & j - i = 1,\ i = 2, 3, \ldots, N-2, \\ 1, & |i-j| = 2,\ i \neq 1,\ N-1, \\ 0, & |i-j| > 2, \end{cases}$$

and $\boldsymbol{R}$ is a column vector given by

$$r_1 = (2 - \tfrac{7}{90}h^4 f_0)\alpha_1 - h^2\beta_1 + \frac{h^4}{360}[28g_0 + 245g_1 + 56g_2 + g_3],$$

$$r_2 = \frac{h^4}{6}[g_1 + 4g_2 + g_3] - \alpha_1,$$

$$r_i = \frac{h^4}{6}[g_{i-1} + 4g_i + g_{i+1}], \qquad i = 3(1)N-3,$$

$$r_{N-2} = \frac{h^4}{6}[g_{N-3} + 4g_{N-2} + g_{N-1}] - \alpha_2,$$

$$r_{N-1} = (2 - \tfrac{7}{90}h^4 f_N)\alpha_2 - h^2\beta_2 + \frac{h^4}{360}[g_{N-3} + 56g_{N-2} + 245g_{N-1} + 28g_N].$$

Now y_i, $i = 1(1)N-1$ can be solved easily from the set of equations (15) making use of the algorithm [1] for the solution of a five-diagonal system. Similarly, the system of equations for the sixth-order method can be obtained.

3. Convergence of the method

We now prove the convergence of the fourth-order scheme. The error equation of the fourth-order scheme is given by

$$AE = T, \tag{16}$$

where $\boldsymbol{E}$ is the error vector and $\boldsymbol{T}$ is the truncation-error vector of the equations given by

$$\begin{aligned} |t_i| &\le 0.002183 h^8 M_8, \qquad i = 2(1)N-2, \\ |t_i| &\le \frac{241}{60480} h^8 M_8, \qquad i = 1, N-1, \end{aligned} \tag{17}$$

where

$$M_n = \operatorname*{Max}_{[a,b]} |y^{(n)}(x)|.$$

For proving the convergence of the method we need to show that the matrix A is monotone. Let

$$P = \begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & \vdots & \vdots & \vdots & & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}, \quad A_0 = \begin{bmatrix} 5 & -4 & 1 & & & & \\ -4 & 6 & -4 & 1 & & & \\ 1 & -4 & 6 & -4 & 1 & & \\ & \vdots & \vdots & \vdots & \vdots & & \\ & & +1 & -4 & +6 & -4 \\ & & & 1 & -4 & 5 \end{bmatrix},$$

$$B = \begin{bmatrix} \frac{49}{72} & \frac{7}{45} & \frac{1}{360} & & & & \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & & & & \\ & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & & & \\ & \cdot & \cdot & \cdot & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ & & & & \frac{1}{360} & \frac{7}{45} & \frac{49}{72} \end{bmatrix}$$

and $D_1 = h^4 \operatorname{diag}(f_1, f_2, \ldots, f_{N-1})$. Then $A_0 = P^2$ and

$$A = A_0 + BD_1 = P^2 + D, \tag{18}$$

where $D = BD_1$. Since $f(x) \ge 0$ for $x \in [a, b]$, we have $D \ge 0$ and hence $A > A_0$.

Following Usmani and Marsden [6], we have

$$P^2 A^{-1} = [I - DP^{-2}][I + (DP^{-2})^2 + (DP^{-2})^4 + \ldots]. \tag{19}$$

Let $\bar{D}$ be obtained from D by replacing f_n by $f_M = \max_{[a,b]} f(x)$, so that $\bar{D} = h^4 f_M B$. By Gershgorin's theorem all the eigenvalues of B lie inside the circle $|\lambda - \frac{2}{3}| = \frac{1}{3}$. Obviously $\rho(B) \le 1$. Hence

$$\rho(\bar{D}) = h^4 f_M \rho(B) \le h^4 f_M \quad \text{and} \quad \rho(P^{-1}) \le \frac{N^2}{8}.$$

Therefore,

$$\rho(DP^{-2}) \le \rho(D)\rho(P^{-2}) \le \rho(\bar{D})\rho^2(P^{-1}) \le \frac{h^4 f_M N^4}{64}.$$

Hence the series (19) will be convergent if

$$f_M < \frac{64}{(b-a)^4}. \tag{20}$$

The matrix P is monotone [3], and hence $A_0 = P^2$ is also monotone. Now, if

$$P^{-2} > P^{-2}\bar{D}P^{-2}, \tag{21}$$

then $G = P^{-2} - P^{-2}\bar{D}P^{-2}$ is a positive matrix and hence $A^{-1} = GM$, where $M = I +$ a positive matrix, will also be positive. Let $B = \bar{B} + \bar{C}$, where

$$\bar{b}_{ij} = \begin{cases} \frac{2}{3}, & i = j = 1, 2, \ldots, N-1, \\ \frac{1}{6}, & |i-j| = 1, \\ 0, & |i-j| > 1, \end{cases}$$

$$\bar{c}_{11} = \bar{c}_{N-1,N-1} = \tfrac{1}{72}, \quad \bar{c}_{12} = \bar{c}_{N-1,N-2} = -\tfrac{1}{90},$$

$$\bar{c}_{13} = \bar{c}_{N-1,N-3} = \tfrac{1}{360} \text{ and } \bar{c}_{ij} = 0 \text{ otherwise.}$$

Let $P^{-2} = (a^*_{ij})$, $P^{-2}\bar{B}P^{-2} = (b^*_{ij})$ and $P^{-2}\bar{C}P^{-2} = (c^*_{ij})$. We know that ([6])

$$a^*_{ij} = \begin{cases} \dfrac{i(N-j)}{6}\left[2j + \dfrac{1}{N} - \dfrac{i^2+j^2}{N}\right], & i \le j, \\[2ex] \dfrac{j(N-i)}{6}\left[2i + \dfrac{1}{N} - \dfrac{i^2+j^2}{N}\right], & i \ge j. \end{cases}$$

We also find that (b^*_{ij}) is symmetric:

$$b^*_{ij} = \tfrac{2}{3}\sum_{k=1}^{N-1} a^*_{ik}a^*_{kj} + \tfrac{1}{6}\sum_{k=1}^{N-2} a^*_{ik}a^*_{k+1,j} + \tfrac{1}{6}\sum_{k=2}^{N-1} a^*_{ik}a^*_{k-1,j}, \qquad i \ge j,$$

and

$$c^*_{ij} = \frac{a^*_{i1}}{360}(5a^*_{1j} - 4a^*_{2j} + a^*_{3j}) + \frac{a^*_{iN-1}}{360}(a^*_{N-3,j} - 4a^*_{N-2,j} + 5a^*_{N-1,j}).$$

Substituting a^*_{ij} from above and simplifying, we get

$$c^*_{i1} = \frac{a^*_{i1}}{360}, \quad c^*_{iN-1} = \frac{a^*_{iN-1}}{360} \text{ and } c^*_{ij} = 0 \text{ otherwise.}$$

Hence, $c^*_{ij} \ge 0$.

From (21) we deduce that A will be monotone if

$$a^*_{ij} \ge h^4 f_M(b^*_{ij} + c^*_{ij})$$

or,

$$f_M \leq \frac{a^*_{ij}}{h^4(b^*_{ij} + c^*_{ij})} \leq \frac{a^*_{ij}}{h^4 b^*_{ij}}, \tag{22}$$

since $c^*_{ij} \geq 0$. Hence from (20) and (22), we conclude that A is monotone if

$$f_M < \mathrm{Min}\left[\frac{a^*_{ij}}{h^4 b^*_{ij}}, \frac{64}{(b-a)^4}\right]. \tag{23}$$

From the equation (18), we have $\|E\| \leq A^{-1}\|T\|$. Since the matrices A and A_0 are both monotone and $A > A_0$, it follows from the theory of monotone matrices ([3]) that $A^{-1} < A_0^{-1}$, so that (see[6])

$$\|E\| \leq A_0^{-1}\|T\|,$$

$$|e_i| \leq \max_k |t_k| \sum_{j=1}^{N-1} a^*_{ij} \leq \frac{241}{60480} h^8 M_8 \sum_{j=1}^{N-1} a^*_{ij} \leq \frac{241}{3870720} h^4 M_8 (b-a)^4. \tag{24}$$

Hence,

$$\|E\| = O(h^4) \to 0 \text{ as } h \to 0,$$

which proves the convergence of the method and that its order is four. We now summarize the above in the following theorem:

THEOREM. *Let $y(x)$ be the exact solution of the boundary values problem* (1)–(2) *and let y_n, $n = 1(1)N-1$ be the exact solution of the system* (15). *If* **E** *is given by* (16) *and $f(x)$ satisfies* (23), $\|E\| = O(h^4)$ *and satisfies* (24), *the round-off error being neglected.*

4. Numerical results

We have used both the fourth- and sixth-order methods for the solution of the following two problems [6].

i) $y^{\text{iv}} + 4y = 1$,

with the boundary conditions $y(\pm 1) = y''(\pm 1) = 0$.
The exact solution is

$$y(x) = 0.25\left\{1 - 2\frac{(\sin 1 \sinh 1 \sin x \sinh x + \cos 1 \cosh 1 \cos x \cosh x)}{\cos 2 + \cosh 2}\right\}$$

ii) $y^{\text{iv}} + xy = -(8 + 7x + x^3)e^x$

with $y(0) = y(1) = 0$ and $y''(0) = 0$, $y''(1) = -4e$.
The exact solution in this case is $y(x) = x(1-x)e^x$.

TABLE 1

Max $|E|$ *in the solutions of the fourth-order equations for* $h = 2^{-m}$, $m = 2(1)7$

	Fourth-order method		*Sixth-order method*		*Second-order method* [6]	
m	*Variable coefficients Problem* (ii)	*Constant coefficients Problem* (i)	*Variable coefficients Problem* (ii)	*Constant coefficients Problem* (i)	*Variable coefficients Problem* (ii)	*Constant coefficients Problem* (i)
2	0.1459(−4)	0.3906(−6)	0.1164(−4)	0.3110(−6)	0.7160(−2)	0.1289(−2)
3	0.5486(−6)	0.1466(−7)	0.1913(−6)	0.5073(−8)	0.1744(−2)	0.3215(−3)
4	0.2829(−7)	0.7569(−9)	0.3117(−8)	0.8167(−10)	0.4330(−3)	0.8031(−4)
5	0.1671(−8)	0.4472(−10)	0.4983(−10)	0.1302(−11)	0.1081(−3)	0.2007(−4)
6	0.1029(−9)	0.2755(−11)	0.7918(−12)	0.2136(−13)	0.2703(−4)	0.5018(−5)
7	0.6458(−11)	0.1850(−12)	0.6588(−13)	0.1145(−13)	0.6756(−5)	—

We note that the solution of (i) is an even function and we integrated the first problem over the interval [0, 1] with step lengths $h = 2^{-m}$, $m = 2(1)7$ and the maximum absolute errors in each case are given in Table 1. Problem (ii) is solved from $x = 0$ to $x = 1$ with the same step lengths and the maximum absolute errors are also recorded in Table 1. The computations are performed in double precision.

From the table we note that, as expected, the errors produced by the sixth-order formula are smaller than the errors in the fourth-order formula. The results produced by both these methods are superior to the second-order method of Usmani and Marsden [6]. It is further verified from the table that on reducing the step size from h to $h/2$, the maximum absolute error is approximately reduced by $\frac{1}{16}$ in the case of the fourth-order method and $\frac{1}{64}$ in the case of the sixth-order method.

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